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The force on a slender fish-like body

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The force acting on a fish-like body with combined thickness and lifting effects is analysed on the assumption of inviscid flow. A general expression is developed for the pressure force on the body, which is analogous to the momentum-flux analysis for non-lifting bodies in classical hydrodynamics. For bodies with constant volume, the mean drag (or propulsive) force is expressed in terms of a contour integral around the vortex sheet behind the body. Attention is focused on the case of steady-state motion with constant angle of attack, and the induced drag is analysed for finned axisymmetric bodies using the slender-body approximation developed by Newman & Wu (1973). Unlike earlier results of Lighthill (1970), the lift–drag ratio in this case depends on the body thickness.

1. Introduction

Recent applications of slender-body theory to fish-like forms, having both thickness and appended lifting surfaces, have been stimulated not only by interest in fish propulsion but also by the applicability of such a theory to study of sailing yachts and submarines. In all three of these cases the essential feature is the presence of both upstream lifting surfaces (side fins, keel or sailplane, respectively) and downstream appendages (caudal fin or rudder), and as a result of the shedding of vorticity at the upstream trailing edges, there is an interaction between the vortex sheet, the downstream portion of the body and the tail fins. Analyses of this type of flow, within the framework of slender-body theory, have been presented by Lighthill (1970), Wu & Newman (1972) and Newman & Wu (1973). Unlike the earlier references, Newman & Wu (1973) accounted not only for the kinematic interaction of the body, fins and vortex sheets, but also for the dynamic effect of the body thicknesses on the trailing vortices; thus by Kelvin's theorem these vortex filaments must follow the 'stretched-straight' streamlines past the body, allowing for the effects of body thickness. The departure of these vortices from the purely longitudinal arrangement common in linearized thin-wing theory is an essential consequence of the consistent slender-body approach.

Taking these vortex–body interactions into consideration, Newman & Wu (1973) obtained explicit results for the linear lifting flow past axisymmetric bodies with planar fin appendages having abrupt trailing edges, and presented computations for the steady lift force with both symmetrical and asymmetrical fin configurations. Their results can be applied more generally to find the yawing moment and distribution of lift force along the body for more general body motions. However, the drag component of the force was not examined, and this is the purpose of the present paper. In order to evaluate the drag force, we shall first develop, in §2, a general expression for the pressure force acting on a portion of the body surface upstream of some arbitrary transverse plane $x =$ constant, and based upon momentum conservation in a similar manner to the classical analysis for irrotational flow. (Here the flow is rotational only in the thin sheets of trailing vorticity.) The resulting expression for the hydrodynamic pressure force appears to be of interest in its own right, since it is free of any assumptions regarding body shape or linearization of the unsteady body motions. For motions with a mean constant forward velocity parallel to the x axis, the force can be expressed in particular as the sums of two terms: the first term is analogous to the added-mass force in irrotational flow, but with the surface integral (of the product of the velocity potential and the unit normal vector) taken not only over the body surface, but also over the vortex-sheet surfaces upstream of the (Trefftz) plane $x =$ constant; and the second term is a surface integral in this Trefftz plane exterior to the body surface. Taking the Trefftz plane on the body gives the forces acting upstream of the plane and hence permits a determination of the differential lift force distribution, whereas taking the Trefftz plane behind the body tail gives the total force acting on the body.

Finally, if slender-body approximations are introduced, the integral over the Trefftz plane can be reduced to a contour integral around the intersection of this plane with the body and/or wake. The resulting expression for the differential lift force is identical to the results of Newman & Wu (1973), and suggests that, in fact, the only effect of nonlinear unsteady motions on the differential lift force is the departure of the trailing vortex sheets from the stretched-straight plane of symmetry of the body. These nonlinear developments are closely related to the Lagrangian analysis of large amplitude fish motions by Lighthill (1971).

Returning to the linear domain of lateral body motion, we use the results of §2 in conjunction with the velocity potential derived by Newman & Wu (1973), in §3, to determine the mean drag force, assuming steady motion of an axisymmetric body with planar foils and a constant angle of attack. The final results agree with classical slender-body theory and low-aspect-ratio wing theory for the special cases of a body without upstream fins and a planar foil without thickness, respectively. More generally, we find a lift–drag ratio which depends on the body geometry in a more complicated way than the corresponding results of Lighthill (1970); this difference is discussed in §4, and is attributed to the effects of body thickness on vortex-sheet dynamics and, in particular, on the departure of our downwash from the constant value associated with planar low-aspect-ratio lifting surfaces.

2. The hydrodynamic pressure force

We wish to consider the pressure force acting on a fish-like body, having both thickness and appended lifting surfaces, which moves with a constant forward velocity U and performs fairly arbitrary unsteady lateral motions. Cartesian co-ordinates (x, y, z) are employed, and are fixed with respect to the mean position of the body, with the body pointed in the negative-x direction. Hence the flow at large distances ahead of the body, relative to the (x,y,z) co-ordinates, is a streaming flow with velocity U directed in the positive-x direction. Ideal inviscid flow is assumed and described by a velocity potential $\phi(x,y,z,t)$ such that the fluid velocity vector is equal to $\nabla\phi$, and $\nabla^2\phi = 0$ except for a discontinuity across the vortex sheets shed at sharp trailing edges of body fins. Following Newman & Wu (1973), we shall assume that the body thickness is symmetrical about the x, y plane, the fins are situated in this plane and that the lateral (swimming) motion is defined by a prescribed body displacement normal to this plane of the form $z = h(x,t)$. Ultimately, we shall assume that this displacement is a small perturbation of the steady 'stretched-straight' flow $h = 0$, and we shall also assume that the body–fin configuration as well as the displacement $h(x,t)$ are slowly varying (i.e. slender) in the x direction, but in the present section a general analysis is made of the pressure force on the body which assumes only that the flow is inviscid and irrotational, except for thin vortex sheets on which the pressure is continuous.

The pressure force $\mathbf{F}$ acting on the portion S_B of the body surface upstream of an arbitrary transverse plane $x =$ constant is, from Bernoulli's equation,

$$\mathbf{F} = -\rho\iint_{S_B}[\phi_t+\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}U^2]\,\mathbf{n}\,dS, \tag{1}$$

where $\mathbf{n}$ is the unit normal vector pointing out of the fluid domain. In order to express this pressure integral in a different form, we define the closed surface $S_B+S_W+S_T+S_\infty$, as shown in figure 1, where S_W is the portion of the trailing vortex sheet upstream of S_T, S_T is the transverse (Trefftz) plane $x =$ constant, exterior to the body, and S_∞ is a fixed closing surface at an infinitely large distance from S_B+S_W. Noting that on S_W the pressure is continuous, the integral in (1) can be extended to the surface S_B+S_W, and hence from Gauss's theorem

$$\mathbf{F} = \rho\iint_{S_T+S_\infty}[\phi_t+\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}U^2]\,\mathbf{n}\,dS-\rho\iiint\nabla[\phi_t+\tfrac{1}{2}(\nabla\phi)^2]\,d\tau, \tag{2}$$

where the volume integral is over the fluid domain bounded by $S_B+S_W+S_T+S_\infty$. Next, we use the transport theorem for the time derivative of a volume integral:

$$\frac{d}{dt}\iiint f\,d\tau = \iiint f_t\,d\tau+\iint f\,U_n\,dS, \tag{3}$$

where U_n is the normal velocity of the boundary surface. Since S_T+S_∞ have zero normal velocity, relative to the (x,y,z) co-ordinate system, and S_B+S_W are material surfaces moving with normal velocity $\partial\phi/\partial n$, it follows from (2) and (3) that

$$\mathbf{F} = \rho\iint_{S_T+S_\infty}[\phi_t+\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}U^2]\,\mathbf{n}\,dS-\rho\frac{\partial}{\partial t}\iiint\nabla\phi\,d\tau+\rho\iint_{S_B+S_W}\phi_n\nabla\phi\,dS$$
$$-\tfrac{1}{2}\rho\iiint\nabla(\nabla\phi)^2\,d\tau. \tag{4}$$

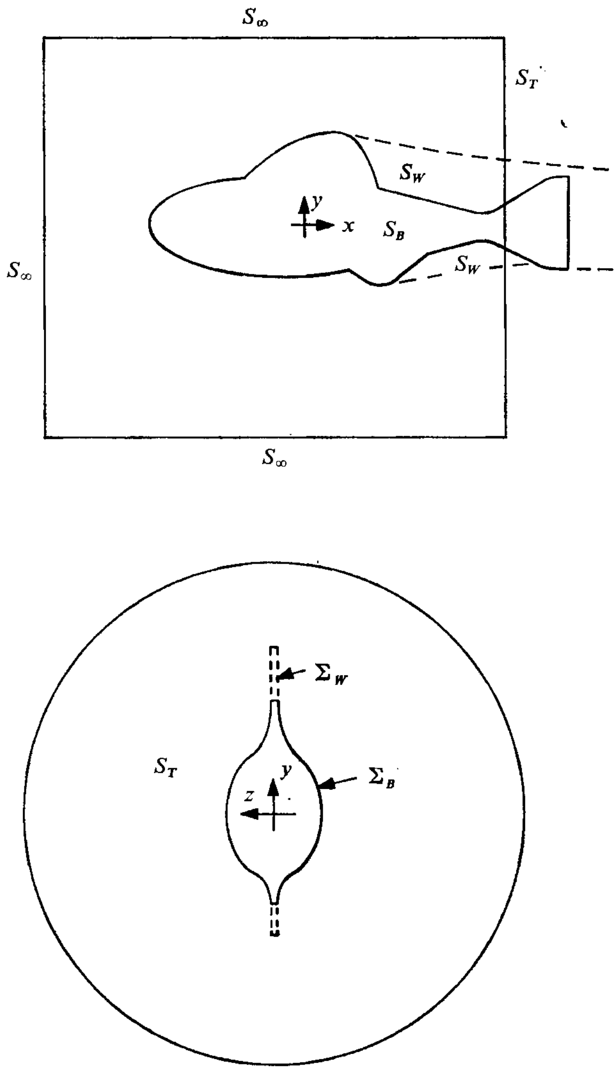

FIGURE 1. Control surfaces for the analysis of the hydrodynamic pressure force.

If Gauss's theorem is applied, noting that on the fixed surface S_∞ the order of time differentiation and space integration may be interchanged, equation (4) takes the form

$$\mathbf{F} = \tfrac{1}{2}\rho \iint_{S_T+S_\infty} [(\nabla\phi)^2 - U^2]\,\mathbf{n}\,dS - \rho\frac{\partial}{\partial t}\iint_{S_B+S_W+S_T} \phi\mathbf{n}\,dS + \rho\iint_{S_B+S_W} \phi_n \nabla\phi\,dS$$
$$-\tfrac{1}{2}\rho\iiint \nabla(\nabla\phi)^2\,d\tau + \rho\iint_{S_T}\phi_t\mathbf{n}\,dS$$
$$= \rho\iint_{S_T+S_\infty}[\tfrac{1}{2}(\nabla\phi)^2\,\mathbf{n} - \tfrac{1}{2}U^2\mathbf{n} - \phi_n\nabla\phi]\,dS - \rho\frac{\partial}{\partial t}\iint_{S_B+S_W+S_T}\phi\mathbf{n}\,dS$$
$$+\rho\iint_{S_T}\phi_t\mathbf{n}\,dS, \quad (5)$$

where Laplace's equation $\nabla^2\phi = 0$ has been invoked. To evaluate the time derivatives of the integral over S_T, we note that S_T is the portion of the *fixed* plane x = constant bounded by the *fixed* curve Σ_∞, where S_T and S_∞ intersect, and the *moving* curve $\Sigma_B + \Sigma_W$, where S_T and $S_B + S_W$ intersect. Thus, by analogy with the transport theorem (3)

$$\frac{\partial}{\partial t}\iint_{S_T} \phi\, dS = \rho \iint_{S_T} \phi_t\, dS + \rho \oint_{\Sigma_B+\Sigma_W} \phi u_n\, dl, \tag{6}$$

where u_n is the normal velocity of Σ_W in the plane x = constant. Substituting in (5), it follows that

$$\mathbf{F} = \rho \iint_{S_T+S_\infty} [\tfrac{1}{2}(\nabla\phi)^2\,\mathbf{n} - \tfrac{1}{2}U^2\mathbf{n} - \phi_n \nabla\phi]\, dS - \rho \frac{\partial}{\partial t}\iint_{S_B+S_W} \phi\mathbf{n}\, dS - \rho\mathbf{i}\oint_{\Sigma_B+\Sigma_W} \phi u_n\, dl, \tag{7}$$

where $\mathbf{i}$ is the unit vector parallel to the x axis.

Equation (7) is quite general, and may be used to find the vector force $\mathbf{F}$, due to the action of hydrodynamic pressures on the portion of the body upstream of the plane S_T, for arbitrary unsteady motion of the body. We shall use this equation first to find the differential lift force $\mathscr{L}(x,t)$ acting on a body cross-section in the direction parallel to the lateral (z) axis, and then to find the mean total drag or propulsive force acting on the entire body, in the longitudinal (x) direction. In both cases the body volume is assumed to be constant, so that for large radial distances r away from the body surface and wake the potential takes the form

$$\phi = Ux + O(1/r^2) \quad \text{as} \quad r \to \infty. \tag{8}$$

Focusing our attention first on the lateral force F_z, acting on that portion of the body situated upstream of the plane x = constant, and using (8) in (7), we see that there is no contribution from S_∞ and it follows that

$$F_z = -\rho \iint_{S_T} \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\, dS - \rho\frac{\partial}{\partial t}\iint_{S_B+S_W} \phi n_z\, dS. \tag{9}$$

This equation is exact, for arbitrary body shapes and motions, but if the additional assumption is made that the body is slender, then to leading order $\phi_x = U$, and hence from Stokes' theorem the lateral force takes the form

$$\begin{aligned} F_z &= -\rho U \oint_{\Sigma_B+\Sigma_W} \phi n_z\, dl - \rho\frac{\partial}{\partial t}\iint_{S_B+S_W} \phi n_z\, dS \\ &= \int_{-l_n}^{x} \mathscr{L}(x,t)\, dx, \end{aligned} \tag{10}$$

where
$$\mathscr{L}(x,t) = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\oint_{\Sigma_B+\Sigma_W} \phi n_z\, dl. \tag{11}$$

This equation for the differential lift force is identical to that obtained by Newman & Wu (1973, equation 6.8), the latter being derived on the basis of a two-dimensional analysis in the cross-flow plane using slender-body approximations from the outset.

In an analogous manner the total drag force is obtained, as the x component of (7), in the form

$$D = \tfrac{1}{2}\rho \iint_{S_T} (U^2 - \phi_x^2 + \phi_y^2 + \phi_z^2)\, dS - \rho \frac{d}{dt} \iint_{S_B+S_W} \phi n_x \, dS - \rho \oint_{\Sigma_W} \phi u_n \, dl, \quad (12)$$

where S_T is taken downstream of the body tail. If the slenderness assumption is invoked (or S_T taken sufficiently far downstream), the velocity ϕ_x will differ from U by a negligible amount on S_T, and hence

$$\begin{aligned} D &= \tfrac{1}{2}\rho \iint_{S_T} (\phi_y^2 + \phi_z^2)\, dS - \rho \frac{d}{dt} \iint_{S_B+S_W} \phi n_x \, dS - \rho \oint_{\Sigma_W} \phi u_n \, dl \\ &= \tfrac{1}{2}\rho \oint_{\Sigma_W} \phi(\phi_n - 2u_n)\, dl - \rho \frac{d}{dt} \iint_{S_B+S_W} \phi n_x \, dS, \end{aligned} \quad (13)$$

where Stokes's theorem and the (two-dimensional) Laplace equation have been used. If we restrict our attention to the mean drag, for oscillatory or steady motion, the time derivative in (13) will not contribute to the average over long time periods, and thus

$$\bar{D} = \tfrac{1}{2}\rho \oint_{\Sigma_W} \overline{\phi(\phi_n - 2u_n)}\, dS, \quad (14)$$

where a bar denotes the time average. Finally, if we now invoke the linearization assumption that the vortex sheets are planar, it follows that, on the upper or lower sides of Σ_W, $\phi_n = \mp \phi_z$ and $u_n = \mp h_t$, where $z = h(x, t)$ is the position of Σ_W. Thus, from (14),

$$\bar{D} = -\tfrac{1}{2}\rho \int_{-a_1}^{a_2} \overline{[\phi]\,(\phi_z - 2h_t)}\, dy, \quad (15)$$

where $(-a_1, a_2)$ is the segment occupied by the contour Σ_W and

$$[\phi] \equiv \phi(y, 0+, t) - \phi(y, 0-, t) \quad (16)$$

is the jump in the potential across the sheet. If the motion is steady, $h_t = 0$ and (15) reduces to the well-known induced-drag integral

$$D = -\tfrac{1}{2}\rho \int_{-a_1}^{a_2} [\phi]\, \phi_z \, dy \quad (17)$$

for steady motion of a three-dimensional planar lifting surface. On the other hand, for unsteady motion of a slender body with a finned tail, having an abrupt trailing edge at the tail $x = l_T$ and no upstream trailing edges, (15) may be applied at $x = l_T$ after noting that

$$\phi_z = V(x, t) \equiv h_t + U h_x \quad (18)$$

and

$$[\phi] = -2V[(y + a_1)(y - a_2)]^{\frac{1}{2}}, \quad (19)$$

whence, for this special case, (15) yields the simple result

$$\bar{D} = \frac{\pi}{2}\left(\frac{a_1 + a_2}{2}\right)^2 [\overline{h_t^2} - U^2 \overline{h_x^2}]. \quad (20)$$

This is in agreement with Lighthill's (1960) equation for the mean propulsive force of a slender fish without side fins,

$$P = -\tfrac{1}{2} m(l_T)\, [\overline{h_t^2} - U^2 \overline{h_x^2}], \quad (21)$$

where $m(l_T)$ is the added-mass coefficient of the tail fin.

3. The drag of a slender finned body

In order to apply the drag integral (15) to a specific problem, we use the potential as derived by Newman & Wu (1973, §3) for a slender fish-like body with thickness symmetrical about the x, y plane and with appended fins confined to the same plane. For this configuration the (lifting) potential† at any section $x =$ constant on the body can be expressed as the real part of the complex potential $f = \phi + i\psi$ which takes the general form

$$f = -iV\zeta + i(\eta^2-\beta^2)^{\frac{1}{2}}\left\{V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta)^{\frac{1}{2}}(\eta_1-\eta)}\right\}. \tag{22}$$

(Here we have omitted a term depending only on x and t, which gives no contribution to (15) but ensures that $f \to 0$ at infinity.) In this formula ζ is the complex variable $\zeta = y+iz$; η is the mapped variable $\eta = \eta_r + i\eta_i$ such that $\eta(\zeta)$ is a conformal transformation of the physical ζ plane into the η plane with $\eta \to \zeta$ at infinity and the body contour mapped to the symmetric cut $\eta_i = 0$, $-\beta < \eta_r < \beta$. The parameters α_1 and α_2 define the outer limits of the vortex sheets in the η plane, so that the trailing vortices shed from the body map into the two segments $(-\alpha, -\beta)$ and (β, α_2). Two special cases are the symmetric configuration where $\alpha_1 = \alpha_2$, and the single side fin and vortex sheet, where only one of the above segments exists, and hence $\alpha_1 = \beta$ or $\alpha_2 = \beta$ for the case of no lower or upper vortex sheets, respectively. With these definitions (and the convention $\alpha_i = \beta$ if no vortex sheets are present), equation (22) is quite general and holds for all regions of the flow, including both leading- and trailing-edge regions of the body as well as the wake downstream of the body.

The remaining parameter in (22) is the vortex-wake potential $\phi_*(\eta_1, x, t)$, which must be constant along the (curved) trailing vortices, when viewed in a fixed reference frame, and hence equal to the value of the potential at the upstream end of the vortex filament at a suitable retarded time. This value, in turn, is governed by the Kutta condition at the trailing edges, and for the case of an abrupt trailing edge parallel to the y axis, it follows from continuity that ϕ_* must equal the value of ϕ immediately upstream of the trailing edge. For a body with a single side fin or a pair of side fins having trailing edges at the same longitudinal position,

$$\phi_* = -V(x_*, t_*)\,(\beta_*^2 - \eta_*^2)^{\frac{1}{2}}. \tag{22}$$

Here x_* is the position of the trailing edges, $\beta_* \equiv \beta(x_*-0)$, t_* is the retarded time

$$t_* = t-(x-x_*)/U \tag{24}$$

and η_* is the value of η on the trailing edge where the vortex filament is shed. Equation (23) is valid for all positions at or upstream of the tail trailing edge $x = l_T$.

† The total potential will be of the form $\phi = Ux + \phi_0 + \phi_1 + \phi_2 + \ldots$, where $Ux + \phi_0$ is the solution of the thickness problem, i.e. the flow past the stretched-straight body with $h = 0$. The potentials $\phi_n = O(h^n)$, so that ϕ_1 is the first-order lifting potential, given in complex form by (22), and ϕ_2, etc., denote nonlinear terms.

For axisymmetric bodies with various fin configurations, Newman & Wu (1973) used this solution in the special case of steady motion at a constant angle of attack V/U, to compute the total lift force. Here we shall carry out the corresponding computations, but using (17) to compute the drag force. For this purpose it is convenient to carry out the integration in (17) at the trailing edge $x = l_T$, where $-\beta < \eta < \beta$ corresponds to the tail trailing edge and the remaining contributions are due to the vortex-sheet segments $-\alpha_1 < \eta < -\beta$ and $\beta < \eta < \alpha_2$ outboard of the fin. Thus we divide (17) into two parts, and use the following relations:

$$\phi_z = V, \tag{25}$$

$$[\phi] = -2(\beta^2-\eta^2)^{\frac{1}{2}}\left\{V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_* \, d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right\}$$

$$\text{on} \quad |\eta| < \beta \quad \text{(tail-fin trailing edge)}, \tag{26}$$

$$\phi_z = -\psi_\eta = \frac{\partial}{\partial\eta}\left\{V\eta - (\eta^2-\beta^2)^{\frac{1}{2}}\left[V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_* \, d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right]\right\}, \tag{27}$$

$$[\phi] = 2\phi_* \quad \text{on} \quad |\eta| > \beta \quad \text{(vortex sheets outboard of tail fin)}. \tag{28}$$

Equation (25) follows from the kinematic boundary condition on the tail fin (and continuity of ϕ_z across the trailing edge); (26) from direct evaluation of (22); (27) from the Cauchy–Riemann equations and (22); and (28) from the fact that $\phi = \pm\phi_*$ on the two sides of the vortex sheet. On substitution of (25)–(28) in (17), the drag force for steady motion is obtained in the form

$$\begin{aligned} D &= -\tfrac{1}{2}\rho\int_{-\alpha_2}^{\alpha_1}[\phi]\,\phi_z\,d\eta \\ &= -\rho\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)d\eta\,\phi_*(\eta)\frac{\partial}{\partial\eta}\left\{V\eta - (\eta^2-\beta^2)^{\frac{1}{2}}\left[V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right]\right\} \\ &\quad + \rho V\int_{-\beta}^{\beta}(\beta^2-\eta^2)^{\frac{1}{2}}\left[V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right]. \end{aligned} \tag{29}$$

Changing the order of integration in the last double integral and integrating with respect to η, it follows that

$$\begin{aligned} \bar{D} &= -\rho\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)d\eta\phi_*(\eta)\frac{\partial}{\partial\eta}\left\{V\eta - (\eta^2-\beta^2)^{\frac{1}{2}}\left[V - \frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right]\right\} \\ &\quad + \rho V\left\{\frac{\pi}{2}V\beta^2 - \left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\phi_*(\eta_1)\left[\frac{\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}} - 1\right]d\eta_1\right\} \\ &= \frac{\pi}{2}\rho V^2\beta^2 - \frac{1}{\pi}\rho\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)d\eta\phi_*(\eta)\frac{\partial}{\partial\eta}\left\{(\eta^2-\beta^2)^{\frac{1}{2}}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}\right\} \end{aligned} \tag{30}$$

or, after integrating by parts,

$$D = \frac{\pi}{2}\rho V^2\beta^2 + \frac{1}{\pi}\rho\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)d\eta\,(\eta^2-\beta^2)^{\frac{1}{2}}\frac{\partial\phi_*}{\partial\eta}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{\phi_*(\eta_1)\,d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}. \tag{31}$$

Equation (31) gives the drag force on an arbitrary body configuration in terms of the vortex-sheet potential ϕ_*. We recall that, in the mapped η plane, the tail

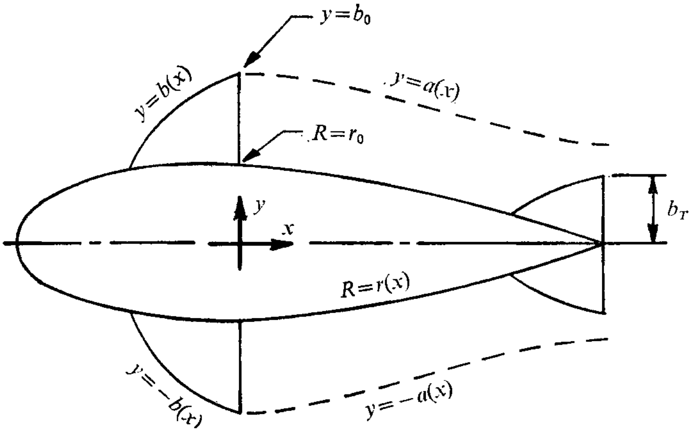

FIGURE 2. Geometrical configuration of the axisymmetric body with symmetric fins.

trailing edge occupies the segment $(-\beta, \beta)$ and the outboard vortex sheets are situated in the segments $(-\alpha_1, -\beta)$ and (β, α_2). For a body with re-entrant tail fins or no outboard vortex sheets, it follows that $\alpha_1 = \alpha_2 = \beta$, and hence from (31) the drag force is equal to $\frac{1}{2}\pi\rho V^2\beta^2$ in accordance with (20). The other special case where simple results are obtainable is that of a planar symmetric foil without thickness where $\alpha_1 = \alpha_2 \equiv \alpha$ and $\phi_* = -V(\alpha^2-\eta^2)^{\frac{1}{2}}$; the resulting double integral is readily evaluated, and it follows that $D = \frac{1}{2}\pi\rho V^2\alpha^2$, where α is the maximum semi-span.

As a more general example, let us consider the axisymmetric body with symmetric fins and abrupt trailing edges, illustrated in figure 2, for which Newman & Wu (1973) computed the lift force. For this case we have (cf. Newman & Wu 1973, equations (7.2)–(7.8)),

$$\begin{aligned}\phi_* &= -V(\beta_*^2-\eta_*^2)^{\frac{1}{2}} \\ &= -V(\alpha^2-\eta^2)^{\frac{1}{2}}(\eta+\alpha^2 r_0^2 b_0^{-2})^{\frac{1}{2}}(\eta^2+r_0^2)^{-\frac{1}{2}}.\end{aligned} \tag{32}$$

where $\alpha_1 = \alpha_2 = \alpha$ and r_0 is the body radius† at the position of the upstream trailing edges, the semi-span of these being b_0. On differentiating (32) with respect to η and using the relation $\alpha^2 = b_0^2 - r_0^2$, it follows that

$$\partial\phi_*/\partial\eta = V\eta^3(\eta^2+2r_0^2)(\alpha^2-\eta^2)^{-\frac{1}{2}}(\eta^2+\alpha^2r_0^2b_0^{-2})^{-\frac{1}{2}}(\eta^2+r_0^2)^{-\frac{3}{2}}. \tag{33}$$

Substituting these expressions in (31) and changing the variables of integration to $u = \eta^2$ and $v = \eta_1^2$, it follows that

$$D = \frac{\pi}{2}\rho V^2\beta^2 - \frac{1}{\pi}\rho V^2 \int_{\beta^2}^{\alpha^2} du \frac{(u-\beta^2)^{\frac{1}{2}}u(u+2r_0^2)}{(\alpha^2-u)^{\frac{1}{2}}(u+\alpha^2r_0^2b_0^{-2})^{\frac{1}{2}}(u+r_0^2)^{\frac{3}{2}}} \times \int_{\beta^2}^{\alpha^2} dv \frac{(\alpha^2-v)^{\frac{1}{2}}(v+\alpha^2r_0^2b_0^{-2})^{\frac{1}{2}}}{(v+r_0^2)^{\frac{1}{2}}(v-\beta^2)^{\frac{1}{2}}(v-u)}. \tag{34}$$

† For an axisymmetric body this radius r_0 is the only parameter associated with body thickness which affects the total lift and drag forces, for steady motion with constant angle of attack.

A more convenient form computationally is obtained by the substitution

$$\frac{(v+\alpha^2 r_0^2 b_0^{-2})^{\frac{1}{2}}}{(v+r_0^2)^{\frac{1}{2}}} \equiv F(v) = F(v)-F(u)+F(u). \tag{35}$$

Then, using the Hilbert transform

$$\int_{\beta^2}^{\alpha^2} \frac{(\alpha^2-v)^{\frac{1}{2}}\,dv}{(v-\beta^2)^{\frac{1}{2}}(v-u)} = -\pi, \tag{36}$$

it follows that

$$\begin{aligned} D = {} & \frac{\pi}{2}\rho V^2\beta^2 + \rho V^2 \int_{\beta^2}^{\alpha^2} du \frac{(u-\beta^2)^{\frac{1}{2}}}{(\alpha^2-u)^{\frac{1}{2}}}\frac{u(u+2r_0^2)}{(u+r_0^2)^2} \\ & -\frac{1}{\pi}\rho V^2 \int_{\beta^2}^{\alpha^2} du \frac{(u-\beta^2)^{\frac{1}{2}} u(u+2r_0^2)}{(\alpha^2-u)^{\frac{1}{2}}(u+\alpha^2 r_0^2 b_0^{-2})^{\frac{1}{2}}(u+r_0^2)^{\frac{3}{2}}} \\ & \times \int_{\beta^2}^{\alpha^2} dv \frac{(\alpha^2-v)^{\frac{1}{2}}}{(v-\beta^2)^{\frac{1}{2}}(v-u)}\left[\left(\frac{v+\alpha^2 r_0^2 b_0^{-2}}{v+r_0^2}\right)^{\frac{1}{2}} - \left(\frac{u+\alpha^2 r_0^2 b_0^{-2}}{u+r_0^2}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{37}$$

The value of the single integral is

$$\frac{\pi}{2}(\alpha^2-\beta^2) - \frac{\pi}{2}\frac{r_0^4(\alpha^2-\beta^2)}{(r_0^2+\beta^2)^{\frac{1}{2}}(r_0^2+\alpha^2)^{\frac{3}{2}}},$$

and hence

$$\begin{aligned} D = {} & \frac{\pi}{2}\rho V^2\alpha^2 - \frac{\pi}{2}\rho V^2 \frac{r_0^4(\alpha^2-\beta^2)}{(r_0^2+\beta^2)^{\frac{1}{2}}(r_0^2+\alpha^2)^{\frac{3}{2}}} \\ & -\frac{1}{\pi}\rho V^2 \int_{\beta^2}^{\alpha^2} du \frac{(u-\beta^2)^{\frac{1}{2}} u(u+2r_0^2)}{(\alpha^2-u)^{\frac{1}{2}}(u+\alpha^2 r_0^2 b_0^{-2})^{\frac{1}{2}}(u+r_0^2)^{\frac{3}{2}}} \\ & \times \int_{\beta^2}^{\alpha^2} dv \frac{(\alpha^2-v)^{\frac{1}{2}}}{(v-\beta^2)^{\frac{1}{2}}(v-u)}\left[\left(\frac{v+\alpha^2 r_0^2 b_0^{-2}}{v+r_0^2}\right)^{\frac{1}{2}} - \left(\frac{u+\alpha^2 r_0^2 b_0^{-2}}{u+r_0^2}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{38}$$

The corresponding formula for the lift force is given by (Newman & Wu, 1973, equation 7.9)

$$L = -\pi\rho U V\beta^2 - 2\rho U V \int_{\beta^2}^{\alpha^2} du \frac{(\alpha^2-u)^{\frac{1}{2}}(u+\alpha^2 r_0^2 b_0^{-2})^{\frac{1}{2}}}{(u-\beta^2)^{\frac{1}{2}}(u+r_0^2)^{\frac{1}{2}}}. \tag{39}$$

Asymptotic approximations for (38) and (39) can be derived for the cases of small body radius ($r_0 \ll \alpha, \beta$) and of nearly re-entrant tail fins ($\alpha^2 \approx \beta^2$). In the former case it follows that

$$D = \frac{\pi}{2}\rho V^2\alpha^2\left[1 - 2r_0^4\left(\frac{\alpha-\beta}{\alpha^4\beta}\right)\right] + O(r_0^6),$$

$$L = -\pi\rho U V\alpha^2\left[1 - r_0^4\left(\frac{\alpha-\beta}{\alpha^4\beta}\right)\right] + O(r_0^6)$$

whereas, for $\alpha^2 \approx \beta^2$,

$$D = \frac{\pi}{2}\rho V^2\beta^2\left[1 + \frac{\beta^2+2r_0^2}{(\beta^2+r_0^2)^2}(\alpha^2-\beta^2)\right] + O(\alpha^2-\beta^2)^2,$$

$$L = -\pi\rho U V\beta^2\left[1 + \frac{(\beta^2+2r_0^2)^{\frac{1}{2}}}{(\beta^2+r_0^2)\,\beta}(\alpha^2-\beta^2)\right] + O(\alpha^2-\beta^2)^2.$$

We note that in both cases the lift–drag ratio depends on the geometric parameters (α, β, r_0).

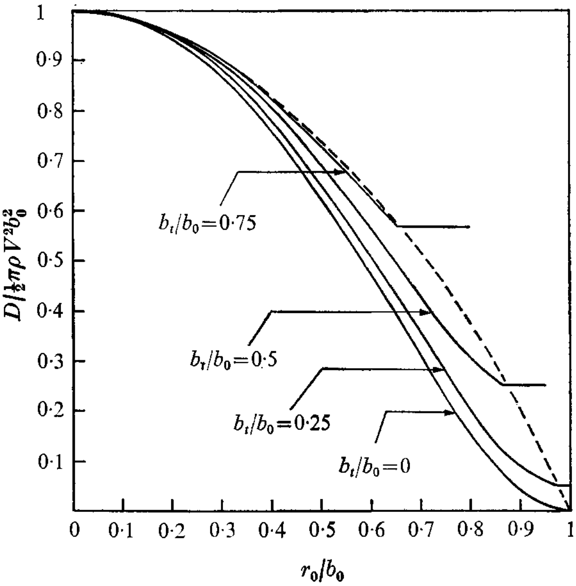

FIGURE 3. Drag coefficient of the axisymmetric body with symmetric fins shown in figure 2, as computed from equation (38). The dashed envelope is the re-entry point where the tail span equals the width of the vortex sheet, and for larger values of b_T the drag coefficient is independent of b_0 and r_0 as shown.

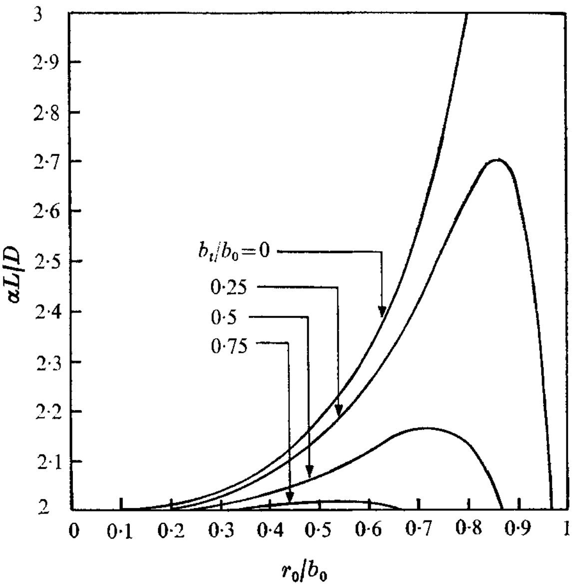

FIGURE 4. The lift–drag ratio times angle of attack corresponding to figure 3 and based on the lift calculations of Newman & Wu (1973, figure 4).

More general values of the drag force require a numerical evaluation of the double integral (38) and are displayed in figure 3, with corresponding values of the lift–drag ratio shown in figure 4. The drag coefficient $D/\frac{1}{2}\pi\rho V^2 b_0^2$ is equal to unity for a planar foil, and hence also when the ratio r_0/b_0 of body radius to maximum semi-span vanishes. For increasing values of the body radius, the drag coefficient falls monotonically, ultimately reaching a value of zero for the (non-lifting) case $r_0/b_0 = 1$. The form of these curves is strikingly similar to the corresponding lift coefficients plotted by Newman & Wu (1973, figure 4). In particular, the drag force depends primarily on the ratio r_0/b_0, and is not so strongly influenced by the tail semi-span b_T, although the drag does increase moderately with increasing values of the ratio b_T/b_0. When the tail span exceeds the span of the vortex sheet shed from the upstream fins, the tail leading edges are 're-entrant' and as in the case of the lift, the drag no longer depends on the body radius or upstream span b_0 (cf. the statement following equation 31). The lift–drag ratio is shown in figure 4, where the ratio $\alpha L/D$ is plotted so as to be independent of the angle of attack $\alpha = V/U$. For a planar foil the ratio $\alpha L/D = 2$, and we see that the effects of body thickness are to increase the lift–drag ratio relative to that of a planar foil with the same plan-form. For the body without tail fins ($b_T/b_0 = 0$), $\alpha L/D$ increases without limit as $r_0/b_0 \to 1$, as a result of the fact that the drag vanishes more rapidly than the lift in this limit; for bodies with tail fins a maximum lift–drag ratio is obtained at an intermediate value of r_0/b_0, and at the re-entrant point the planar value of two is again obtained, in consequence of the fact that both lift and drag then depend only on the tail span.

4. Discussion

As noted in the introduction, the present computations of the induced drag, and the underlying solution of Newman & Wu (1973) for the boundary-value problem of a finned slender body with outboard vortex sheets, differ from the earlier studies of Lighthill (1970), and Wu & Newman (1972), by the consideration of the effects of body thickness on the dynamics of the trailing vortex sheets. The quantitative importance of this effect is not evident *a priori*, nor are calculations of the lift and drag based on the earlier theories available for comparison. However, some insight into this question can be gained by outlining the approach of Lighthill (1970) and focusing on the difference between our respective values for the ratio of total lift force to drag, in the special case of steady-state motion with constant angle of attack. Thus, we write the velocity potential upstream of the first trailing edge $x = x_*$ in the form $\phi = V\varphi$ and downstream of $x = x_*$ in the form $\phi = V(\varphi + \tilde{\varphi})$, where $V =$ constant. The 'leading-edge' potential φ satisfies the boundary condition $\partial\varphi/\partial n = n_z$ on the body contour Σ_B and $\varphi = 0$ on the wake contour Σ_W. The potential $\tilde{\varphi}$ accounts for the presence of trailing vortices on Σ_W and satisfies the boundary condition $\partial\tilde{\varphi}/\partial n = 0$ on Σ_B. In this way the velocity potential is decomposed, at each section of the body, into a part due to the body motions in irrotational flow and a part due to the outboard vortex sheets in the presence of a *stationary* body. (In the notation of Newman & Wu (1973), $V\varphi$ and $V\tilde{\varphi}$ are given by the real parts of the analytic functions f_e

and f_c, respectively.) The potential $\tilde{\varphi}$ is uniquely determined by the Kutta condition

$$\lim_{x\to x_*-0} \varphi = \lim_{x\to x_*+0} (\varphi+\tilde{\varphi}). \tag{40}$$

The virtual mass $m(x)$ is defined in the conventional manner, in terms of the potential φ, in the form

$$m = \rho \oint_{\Sigma_B} \varphi n_z \, dl = \rho \oint_{\Sigma_B+\Sigma_W} \varphi \frac{\partial \varphi}{\partial n} dl \tag{41}$$

and by analogy, the virtual mass $\tilde{m}(x)$ is defined as

$$\tilde{m} = \oint_{\Sigma_B+\Sigma_W} \tilde{\varphi} \frac{\partial \tilde{\varphi}}{\partial n} dl. \tag{42}$$

From Green's theorem and the boundary conditions on Σ_b, it follows that

$$\oint_{\Sigma_B+\Sigma_W} \tilde{\varphi} \frac{\partial \varphi}{\partial n} dl = \oint_{\Sigma_B+\Sigma_W} \varphi \frac{\partial \tilde{\varphi}}{\partial n} dl = 0, \tag{43}$$

and hence the total kinetic energy in the cross-flow plane is equal to

$$\tfrac{1}{2}\rho \int_{\Sigma_B+\Sigma_W} \phi \frac{\partial \phi}{\partial n} dl = \tfrac{1}{2} V^2 (m+\tilde{m}). \tag{44}$$

On the other hand, the lateral momentum in the same plane of fluid is

$$\rho \oint_{\Sigma_B+\Sigma_W} \phi n_z \, dl$$

and *if the downwash behind the trailing edge is constant*, this momentum is equal to

$$\frac{\rho}{V} \oint_{\Sigma_B+\Sigma_W} \phi \frac{\partial \phi}{\partial n} dl = V(m+\tilde{m}), \tag{45}$$

where the kinematic boundary condition at $x = x_*$ has been used to replace n_z on Σ_W by $\partial\phi/\partial n$.

It now follows, by conservation of energy and (44), that the induced drag is equal to

$$D = \tfrac{1}{2} V^2 (m+\tilde{m})_{x=l_T} \tag{46}$$

whereas, from conservation of lateral momentum and (45), the total lift force is given by the expression

$$L = UV(m+\tilde{m})_{x=l_T}. \tag{47}$$

Equations (46) and (47), which are special cases of Lighthill's (1970) propulsive force (equation (27)) and (integrated) differential lift force (equation (25)), respectively, indicate, in particular, that the lift–drag ratio is equal to $2U/V$ and is independent of the body geometry. On the other hand, the computations shown in figure 4 do not agree with this value and, indeed, depend on the body thickness and fin spans. The reason for this discrepancy appears to lie with the assumption, in (45), that the downwash is constant on Σ_W. Indeed, the downwash on Σ_W may be computed from (27), and is equal to V at $x = x_*$, but departs from this constant value downstream owing to the dependence of ϕ_* on x.

In conclusion, we may regard the values of $\alpha L/D$ in figure 4 as a measure of the importance of the vortex–body interactions emphasized by Newman & Wu (1973). For small values of r_0/b_0, the effects of body thickness are small and hence the departure of the vortex sheets from their initial state of constant downwash is small, so that by equations (46) and (47) $\alpha L/D \approx 2$. As the ratio r_0/b_0 increases, the interaction between the body thickness and vortex sheets becomes more significant, but ultimately the tail fin dominates both lift and drag (for $b_T/b_0 = 0$), and the interaction between the body and outboard vortices upstream of the tail is then of no significance to the lift–drag ratio. The effects of this interaction for more general unsteady, undulatory motions, as in fish propulsion, cannot be estimated quantitatively from these conclusions; however, the curve $b_T/b_0 = 0$ in figure 4 may constitute an approximate upper bound for the effect on the general steady-state lift distribution $\mathscr{L}(x)$, since the tail fin does not affect the upstream flow. For undulatory motions, in space and/or time, it seems likely that phase differences of the vorticity in the streamwise direction will tend to dominate the interaction effect so that figure 4 may also serve as an upper-bound estimate in the case of more general body motions. Assuming these arguments are valid, the error in neglecting the vortex–body interactions, as in the earlier work of Lighthill (1970) and Wu & Newman (1972), should be no more than 10 % for $r_0/b_0 < 0{\cdot}5$, or a fish with side-fin span greater than twice the body depth. Finally, we emphasize that the above conclusions are based on the assumption of an axisymmetric body; for bodies which are flattened either in the vertical or lateral direction, one should approach the planar foil situation with trailing vortices parallel to the undisturbed free stream, and hence the interaction effect should be still smaller than the above estimates. In the absence of more detailed numerical calculations, however, these comments are somewhat speculative.

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